# Resonant diffusion in a linear network of fluctuating obstacles

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In this paper we consider the motion of a particle in a linear array of fluctuating barriers. Each barrier can be closed or open and fluctuates in time. The motion of the particle is any stochastic motion between the barriers and the closed barriers stops the particle. We give several rigorous asymptotic results for the transmission probability of the particle and use them to show that this model presents a stochastic resonance with respect to the probability of finding a barrier closed. [S1063-651X(98)15212-1]

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## INTRODUCTION

The motion of a particle in a disordered environment is usually studied in one of the following situations. Either one assumes that the fluctuations in time of the environment are extremely fast compared to the time scale of the motion of the particle or one assumes that the fluctuations in time of the environment are infinitely slow with respect to the time scales of the motion of the particle. The first situation (fast environment) corresponds to the usual Markovian limit: The particle experiences an environment that essentially has no memory. The second situation (very slow environment) corresponds to a quenched disorder and has recently been the subject of many studies (see [1,2] for reviews and [3] for an application to chemical kinetics in the presence of diffusion).

The intermediate situation where the environment fluctuates at a finite time scale is rather unknown. Recently, the situation of a diffusion in a fluctuating potential (between two states) has been extensively studied (see [4–6]) and phenomena of the resonance have been investigated. For reasons related to chemical kinetics, the case of potentials modeled by two fluctuating barriers has also been studied (see [7]) and a resonance of the transmission probability has been exhibited as a maximum of the transmission probability as a function of the probability to find a barrier closed. In this case, the potential was fluctuating between four states.

The subject of this article is the diffusion of a particle in a disordered environment that is fluctuating in time. The environment is idealized. A particle is moving on a long interval $[0,N]$. At each point $n=0,1,2,\ldots,N$, an impenetrable barrier can appear at random times. In each interval $[n,n+1]$ the particle has a certain given stochastic motion (which can be of any type, even non-Markovian) and when the particle arrives at site $n+1$, say, it continues in $[n+1,n+2]$ if the barrier at $n+1$ does not exist at the instant of arrival and it is reflected back in $[n,n+1]$ if the barrier at $n+1$ is present at the instant of arrival. The barriers at various sites are uncorrelated and the motion of the particle does not influence the environment. Although this model is quite simple, it cannot be solved exactly. Nevertheless, it is possible the find rigorous asymptotic results (in various limits of the parameters controlling the fluctuation of the barriers) for the overall transmission probability, that is, the probability that a particle entering the interval at 0 leaves it at $N$ at a certain time. These asymptotic results are sufficient to prove the existence of a resonance in the following sense. For a certain range of frequencies of the fluctuation of the barrier at each site, the overall transmission probability presents a maximum as a function of the probability to find a barrier closed. This result seems paradoxical in the following sense. At first sight, the presence of the barrier would hinder the motion. However, if the stochastic motion (without barriers) has a sufficiently low probability of transmission this turns out to be wrong. The presence of closed barriers may facilitate the transmission of a particle because it also hinders the return of the particle at its starting point. Previous numerical simulations [8] were not very conclusive, although they tended to confirm the phenomenon (we shall examine reasons why this is so in this article). We notice that this phenomenon was exhibited explicitly for two barriers in [7] by analytical methods.

Although idealized, this kind of model has been used and solved under certain approximations to describe diffusion in glasses [9] and also diffusion of a substrate in enzymatic reactions [10]. In the latter case the diffusion of the substrate is conceived as a traversal of a series of bottlenecks that the structure of the enzyme contains naturally, which can be closed or open according to the conformational fluctuation of the protein. These kinds of phenomena have been experimentally observed [10] or simulated (for example, in myoglobin [11]) and analytically described as a series of two-state barriers [12]. Our result proves, as an unexpected consequence, that a protein can also facilitate the diffusion of a substrate towards the reactive site. In other words, a protein may also very well be, in certain circumstances, a diffusional catalyst as well as having its traditional function of chemical catalyst.

We now describe briefly the content of this article. In Sec. I we return to the two-barrier cases and give more precise results than those previously obtained in [7]. These preliminary results are necessary for Sec. II, where we present the model of diffusion in $N$ barriers and certain asymptotic results. In Sec. III we give another asymptotic result (limit of all barriers open) and in Sec. IV we derive the resonance result. The mathematical calculations and notation are rather cumbersome. They are postponed to Appendixes A–D.

 

## I. ASYMPTOTIC ANALYSIS OF THE TWO-BARRIER SYSTEM

### A. Description of the two-barrier system and notation

We consider first the two-barrier system, namely, we have a certain interval, say [0,1] and a stochastic process in [0,1]. We denote $s(t)dt$ the probability that the particle starting from 0 leaves the interval at 1 in the time interval $[t,t+dt]$ and $r(t)dt$ the probability that the particle starting from 0 leaves the interval at 0 in the time interval $[t,t+dt]$. We assume that the stochastic process is symmetric with respect to the exchange of 0 and 1 and there are no losses inside the interval

$$\int_0^\infty s(t)dt+\int_0^\infty r(t)dt=1.$$

We also denote

$$R=\int_0^\infty r(t)dt, \quad S=\int_0^\infty s(t)dt. \tag{1.1}$$

Otherwise, there are no further assumptions about the stochastic process (which can even be non-Markovian).

Now we add two barriers at 0 and 1. These barriers can be in two states labeled $\epsilon=0,1$. When a particle hits a barrier in a closed state $(\epsilon=1)$, it starts its stochastic process afresh and is reflected back in the interval. When it hits a barrier in an open state $(\epsilon=0)$ it leaves the interval. The probability $\varphi_{\epsilon'\epsilon}(t)$ of finding a barrier in state $\epsilon'$ at time $t$ while it is in state $\epsilon$ at time 0 is given by the dichotomous noise law

$$\varphi_{\epsilon'\epsilon}(t)=\alpha_{\epsilon'}+(\delta_{\epsilon\epsilon'}-\alpha_{\epsilon'})e^{-\lambda t}, \tag{1.2}$$

where $\alpha_0,\alpha_1$ are the stationary probabilities of the states 0 and 1 of the barrier, so that $\alpha_0+\alpha_1=1$, and λ is the fluctuation frequency.

We now want to compute the various transmission and reflection probabilities of the whole system (the stochastic process of the particle in the presence of the fluctuating barriers). Unfortunately, the notation becomes rather prohibitive. We shall denote $s_{a'a}(t;\{\epsilon'\}|\{\epsilon\})dt$ the probability that the particle leaves the interval in time $[t,t+dt]$ at point $a'=0,1$ the states of the barrier being $\{\epsilon'_0,\epsilon'_1\}=\{\epsilon'\}$, knowing that it starts at time 0 from point $a=0,1$, the state of the barriers being $\{\epsilon_0,\epsilon_1\}=\{\epsilon\}$. Of necessity, here $\epsilon'_{a'}=0$ because the barrier at $a'$ must be open to let the particle go through the barrier (and out of the interval). We note that

$$s_{a'a}(t|\{\epsilon\})dt=\sum_{\{\epsilon'\}} s_{a'a}(t;\{\epsilon'\}|\{\epsilon\}). \tag{1.3}$$

We find it convenient to denote the Laplace transform of a function $f(t)$, with the Laplace parameter $\theta$ as $f^{(\theta)}$ or $[f]^\theta$,

$$f^{(\theta)}\equiv[f]^\theta\equiv\int_0^\infty e^{-\theta t}f(t)dt,$$

$$[f]\equiv[f]^0 \quad (\text{i.e., } [f]^\theta \text{ with } \theta=0).$$

The reason for these somewhat unusual abbreviations will appear shortly. In particular

$$S^{(\theta)}_{a'a}(\{\epsilon'\}|\{\epsilon\})=\int_0^\infty s_{a'a}(t;\{\epsilon'\}|\{\epsilon\})e^{-\theta t}dt,$$

$$S^{(\theta)}_{a'a}(\{\epsilon\})=\int_0^\infty s_{a'a}(t|\{\epsilon\})e^{-\theta t}dt.$$

When we set $\theta=0$ we simply denote without any upper index

$$S_{a'a}(\{\epsilon'\}|\{\epsilon\})=S^{(0)}_{a'a}(\{\epsilon'\}|\{\epsilon\}).$$

### B. The system for $S^{(\theta)}_{a,0}(\epsilon_0,\epsilon_1)$

We start by writing a system of equations for $S^{(\theta)}_{a,0}(1_0,\epsilon_1)$. This will be a 4×4 closed system. We have

$$S^{(\theta)}_{10}(1_0,\epsilon_1)=[s\varphi_{0\epsilon_1}]^\theta+[s\varphi_{1\epsilon_1}\varphi_{\epsilon'_0 1}]^\theta S^{(\theta)}_{00}(1_0,\epsilon'_0)+[r\varphi_{11}\varphi_{\epsilon'_1\epsilon_1}]^\theta S^{(\theta)}_{10}(1_0,\epsilon'_1), \tag{1.4}$$

$$S^{(\theta)}_{00}(1_0,\epsilon_1)=[r\varphi_{01}]^\theta+[s\varphi_{\epsilon_0 1}\varphi_{1\epsilon_1}]^\theta S^{(\theta)}_{10}(1_0,\epsilon_0)+[r\varphi_{11}\varphi_{\epsilon'_1\epsilon_1}]^\theta S^{(\theta)}_{00}(1_0,\epsilon'_1). \tag{1.5}$$

In this system, which is closed, the convention is that repeated indices in a given monomial are summed over their possibilities 0,1. Let us comment briefly on Eq. (1.4): The full propagator for leaving by 1, starting from 0, the barrier at 0 being closed, and the barrier at 1 being in state $\epsilon_1$ is the sum of the following contributions.

*(i)* $[s\varphi_{0\epsilon_1}]^\theta$. This is the propagator for the direct motion of the stochastic particle from 0 to 1 and when it arrives at 1 it finds the barrier open.

*(ii)* $[s\varphi_{1\epsilon_1}\varphi_{\epsilon'_0 1}]^\theta S^{(\theta)}_{00}(1_0,\epsilon'_0)$. This is the propagator for the direct motion of the stochastic particle from 0 to 1, then finds at 1 a closed barrier, the barrier at 0 is in a certain state $\epsilon'_0$, followed by the full return propagator (from 1 to 1, but recall that our system is symmetric) knowing that the barrier at 1 is closed, and the barrier at 0 is in state $\epsilon'_0$,

$$S^{(\theta)}_{11}(\epsilon'_0,1_1)\equiv S^{(\theta)}_{00}(1_0,\epsilon'_0) \quad (\text{by symmetry}).$$

*(iii)* $[r\varphi_{11}\varphi_{\epsilon'_1\epsilon_1}]^\theta S^{(\theta)}_{10}(1_0,\epsilon'_1)$. This is the propagator for the motion of the particle returning to 0 (before reaching 1) and finding the barrier at 0 closed, the barrier at 1 having fluctuated to state $\epsilon'_1$, followed by the full propagator from 0 to 1, the barrier at 0 being closed, the barrier at 1 being in its new state $\epsilon'_1$. This explains Eq. (1.4) and Eq. (1.5) is explained in a similar way.

Knowing the quantities $S^{(\theta)}_{a0}(1_0,\epsilon_1)$, it is easy to compute all the other quantities $S^{(\theta)}_{a0}(0,\epsilon)$. We have

$$S^{(\theta)}_{00}(0,\epsilon)=[r\varphi_{00}]^\theta+[r\varphi_{10}\varphi_{\epsilon'\epsilon}]^\theta S^{(\theta)}_{00}(1,\epsilon')+[s\varphi_{1\epsilon}\varphi_{\epsilon'0}]^\theta S^{(\theta)}_{10}(1,\epsilon'), \tag{1.6}$$

$$S^{(\theta)}_{10}(0,\epsilon)=[s\varphi_{0\epsilon}]^\theta+[s\varphi_{1\epsilon}\varphi_{\epsilon'0}]^\theta S^{(\theta)}_{00}(1,\epsilon')+[r\varphi_{10}\varphi_{\epsilon'\epsilon}]^\theta S^{(\theta)}_{10}(1,\epsilon') \tag{1.7}$$

(again with summation over respected indices $\epsilon'$ understood).

### C. Asymptotic values of $S_{a0}^{(\theta)}(\{\epsilon\})$: Limit of closed barriers

In this section we consider the asymptotics of the transmission or reflection probabilities $S_{a0}^{(\theta)}(\{\epsilon\})$ in the limit of closed barriers, that is, when $\alpha_0 \to 0$.

#### *1. The case θ>0*

The calculations are given in Appendix A. The quantities $S_{a0}^{(\theta)}(\epsilon_0,\epsilon_1)$ are all 0 when $\epsilon_a=1$ in the limit of $\alpha_0=0$, as results from Eqs. (A11)–(A14). The nonzero quantities $S_{a0}^{(\theta)}(\{\epsilon\})$ are given in these equations (A11)–(A14). In other words, one cannot leave the interval [0,1] by a point where the barrier was initially closed, in any finite time interval.

#### *2. The case θ=0*

In this case we notice that in the limit $\alpha_0=0$,

$$S_{00}(1,1)=S_{10}(1,1)=\tfrac{1}{2}$$

and the other quantities $S_{a0}(\{\epsilon\})$ are given in Eqs. (A16), (A17), and (A19)–(A22).

#### *3. Remark*

We see that $S_{00}^{(\theta)}(1,1)$ or $S_{00}^{(\theta)}(1,0)$ are 0 for $\theta\neq 0$ when $\alpha_0=0$, while they are not 0 for $\theta=0$ and $\alpha_0=0$. This is related to the following phenomena. Suppose we want to compute the transmission probability $S_{01}(0,1)$ using a sum over paths. This in fact can be easily done if the stochastic process in [0,1] is the ballistic motion (i.e., the particle starts from 0, say, and arrives at 1 at a given time $\tau$, its motion being uniform).

It is easy to see that, by summing over all possible paths,

$$S_{01}(0,1)=\varphi_{01}(\tau)+\varphi_{11}(\tau)\varphi_{10}(2\tau)\varphi_0(2\tau)\sum_{n\geqslant 0}\varphi_{11}(2\tau)^{2n}.$$

If we stop the expansion at any finite order, we see that the result is 0 when $\alpha_0$ tends to 0. On the other hand, the geometric series, when summed, gives the factor $[1-\varphi_{11}(2\tau)^2]^{-1}\equiv\varphi_{01}(2\tau)^{-1}[1+\varphi_{11}(2\tau)]^{-1}$ and the $\alpha_0$ in factor of $\varphi_{01}$ cancels with the $\alpha_0$ in the numerator leading to a finite and nonzero result when $\alpha_0$ tends to 0. So we have a situation where at any finite order of perturbation theory, the resulting quantity is zero while the exact result is nonzero.

At the level of the system of equations (A2) and (A3) this phenomenon manifests itself in that the system degenerates for $\theta=0$ and $\alpha_0=0$ and becomes underdetermined in this circumstance. The method to lift the underdetermination is then to use the conservation of probability (the particle must finally leave [0,1]).

### D. Asymptotic values of $S_{a0}^{(\theta)}(\{\epsilon\})$ for small frequencies λ

#### *1. θ>0*

We set $\lambda=0$ in Eqs. (A2) and (A3). In particular $\varphi_{\epsilon'\epsilon}=\delta_{\epsilon'\epsilon}$ and one finds

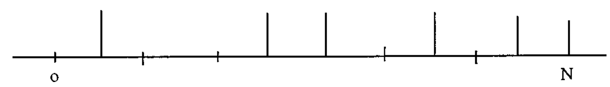


FIG. 1. Linear array of $N$ barriers, with certain barriers open or closed.

$$S_{00}^{(\theta)}(1,0)=S_{00}^{(\theta)}(1,1)=S_{00}^{(\theta)}(1,1)=0,$$
$$S_{00}^{(\theta)}(1,0)=\frac{\hat{s}(\theta)}{1-\hat{r}(\theta)}. \tag{1.8}$$

#### *2. θ=0*

For $\theta=0$, the system of equations (A2) and (A3) becomes degenerate. It is nevertheless possible to show that

$$S_{10}(1,0)=1,\quad S_{00}(1,0)=0,$$
$$S_{00}(1,1)=S_{10}(1,1)=\tfrac{1}{2}, \tag{1.9}$$

from which we deduce, using Eqs. (1.6) and (1.7),

$$S_{00}(0,0)=R,\quad S_{00}(0,1)=1,$$
$$S_{10}(0,0)=S,\quad S_{10}(0,1)=0. \tag{1.10}$$

## II. DIFFUSION IN A LINEAR ARRAY OF FLUCTUATING BARRIERS

### A. Description of the system and notation

We consider now an interval of length $N$ ($N$ is an integer) (see Fig. 1). In each interval $[j,j+1]$, one has a certain stochastic process that is characterized by the following quantities: $s_j(t)dt$ is the probability that the particle starting from $j$, in the interval $[j,j+1]$, leaves the interval through point $j+1$ between times $t$ and $t+dt$ and $r_j(t)dt$ is the probability that the particle starting from $j$ in the interval $[j,j+1]$ leaves the interval through point $j$ between times $t$ and $t+dt$.

We assume that each stochastic process in each interval $[j,j+1]$ is symmetric with respect to the exchange of the extremities of the interval. Moreover, at each point $j$, we place a fluctuating barrier that can be in state $\epsilon_j=0,1$ and jumps between these two states according to Eq. (1.2). Otherwise, we do not assume anything else about the stochastic process.

If the particle arrives at a point $j$ (coming from the left or from the right) when the barrier at $j$ is in the open state $\epsilon_j=0$, the particle enters the next interval (say $[j,j+1]$ or $[j-1,j]$, respectively) and starts a stochastic motion that is independent of the previous stochastic motion. If the particle arrives at a point $j$ (from the right or the left) and the barrier at point $j$ is in the closed state $\epsilon_j=1$, the particle is reflected back in the interval from which it comes (to $[j,j+1]$ or $[j-1,j]$, respectively) and starts again an independent stochastic motion in this interval. Our aim is to study the quantity $S_{[0,N]}(0,\{\epsilon_k\}_{k\geqslant 1})$, which is the probability that the particle leaves the interval $[0,N]$ at $N$, knowing that it has entered the interval at 0 at time 0.

### B. Asymptotics for $\alpha_0=0$

Starting from the system of equations (B2) and (B3), it is possible to derive rigorous asymptotic results for the limit $\alpha_0=0$ (all barriers closed). $S_{[0,N]}(0,\{1\}_{k\geqslant 1})$ is the probability that the particle starting at time $t=0$ from 0 leaves the interval $[0,N]$ by point $N$, the states of the barriers being $\epsilon_0=0$ and $\epsilon_k=1$ ($k\geqslant 1$) at time 0. It is remarkable that one obtains an addition formula for this probability in the form

$$\frac{1}{S_{[0,N]}(0,\{1\}_{k\geqslant 1})}-1=\sum_{j=0}^{N-1}\left(\frac{1}{S_{j+1,j}(0,1)}-1\right). \tag{2.1}$$

We notice also that $S_{[0,N]}(0,\{1\}_{k\geqslant 1})$ is, in the limit $\alpha_0=0$, the transmission probability for a particle entering the system at time 0 at point 0 (with the barrier at 0 open), the other barriers $k=1,\ldots,N$ being in their stationary states. If all the intervals and the stochastic processes are identical, Eq. (2.1) gives

$$S_{[0,N]}(0,\{1\}_{k\geqslant 1})=\frac{S_{10}(0,1)}{N-(N-1)S_{10}(0,1)}, \tag{2.1a}$$

with $S_{10}(0,1)$ given by Eqs. (A20) and (A22),

$$S_{10}(0,1)=\frac{1}{2}\left[1-\hat{r}(\lambda)-\frac{\hat{s}(\lambda)^2}{1-\hat{r}(\lambda)}\right]. \tag{2.1b}$$

### C. Asymptotics of the transmission probability for $\lambda=0$

We consider now the transmission probability $S_{[0,N]}(t;\{\epsilon'\}|\{\epsilon\})dt$, which is the probability starting from 0 (in the system $[0,N]$), with the states $\epsilon_k$ for the barriers, to leave the system $[0,N]$ at point $N$ between times $t$ and $t+dt$ and with the states of the barriers $\epsilon'_k$ (necessarily $\epsilon'_N=0$). In the same manner $S_{[0,N]}(t;\{\epsilon'\}|1^+,\{\epsilon\})dt$ denotes the analogous quantity, but starting from point 1 with positive velocity. The recurrence equations for the total transmission probabilities are given by

$$S_{[0,N]}(\{\epsilon'\}|0,\{\epsilon\}_{k\geqslant 1})=\left[S_{10}(\eta_0,0|0,\epsilon_1)\prod_{k\geqslant 2}\varphi_{\eta_k\epsilon_k}\right]\times S_{[0,N]}[\{\epsilon'\}|1^+;\eta_0,0_1(\eta_k)_{k\geqslant 0}], \tag{2.2}$$

$$\begin{aligned}S_{[0,N]}(\{\epsilon'\}|1^+,\epsilon_0,0_1,\{\epsilon_k\}_{k\geqslant 2})&=[S_{[1,N]}(t,\{\epsilon'\}|0_1,\{\epsilon_k\}_{k\geqslant 2})\varphi_{\epsilon'_0\epsilon_0}]\\&\quad+[R_{[1,N]}(t,\{\eta_k\}_{k\geqslant 1}|0_1,\{\epsilon_k\}_{k\geqslant 2})\varphi_{\eta_0\epsilon_0}]\\&\quad\times\left[S_{00}(t,0_0,\eta'_0|0_0,\eta_0)\prod_{k\geqslant 2}\varphi_{\eta'_k\eta_k}\right]\\&\quad\times S_{[0,N]}(\{\epsilon'\}|1^+;\eta'_0,0_1,\{\eta'_k\}_{k\geqslant 2});\end{aligned} \tag{2.3}$$

this is supplemented by the set of equations (B2) and (B3) for the $R$'s, which are defined in Appendix B. The bracket $[\cdots]$ notations are as in Sec. I A.

When $\lambda$ tends to 0, all the $\varphi_{\epsilon'\epsilon}$ reduce to $\delta_{\epsilon'\epsilon}$. Moreover, the two-barrier quantities $S_{a0}(\{\epsilon'\}|\{\epsilon\})$, $a=0,1$, are equal to $S_{a0}(\{\epsilon\})\delta_{\epsilon'\epsilon}$, where $S_{a0}(\{\epsilon\})$ are given by the formulas of Sec. I D in the limit $\lambda=0$.

We shall assume, as a recurrence hypothesis, that $S_{[0,N-1]}(\{\epsilon'\}|0_0,\{\epsilon\}_{k\geqslant 1})$ is 0 except if $\epsilon_k=0$ for all $k\geqslant 1$ and $\{\epsilon'\}=\{0_0,\{\epsilon\}_{k\geqslant 1}\}$. This is true for the two-barrier system. Now Eq. (2.2) reduces in the limit $\lambda=0$ to

$$S_{[0,N]}(\{0\}_{k\geqslant 0}|\{0\}_{k\geqslant 0})=S_{10}(0,0)S_{[0,N]}(\{0\}_{k\geqslant 0}|1^+;\{0\}_{k\geqslant 0}), \tag{2.4}$$

while in Eq. (2.3), if we want a nonzero $R_{[1,N]}$ and a nonzero $S_{[1,N]}$, we must have $\{\epsilon_k\}=\{\eta_k\}$ and

$$S_{[0,N]}(\{0\}_{k\geqslant 0}|1^+;\{0\}_{k\geqslant 0})=\frac{S_{[1,N]}(\{0\}_{k\geqslant 1}|\{0\}_{k\geqslant 1})}{1-S_{00}(0,0)R_{[1,N]}(\{0\}_{k\geqslant 1}|\{0\}_{k\geqslant 1})}, \tag{2.5}$$

so that from Eqs. (2.4) and (2.5) and from the fact that

$$R_{[1,N]}(\{0\}_{k\geqslant 1}|\{0\}_{k\geqslant 1})=1-S_{[1,N]}(\{0\}_{k\geqslant 1}|\{0\}_{k\geqslant 1}). \tag{2.6}$$

Inserting Eq. (2.6) into Eq. (2.5), an addition formula is easily obtained by induction:

$$\frac{1}{S_{[0,N]}(\{0\}_{k\geqslant 0}|\{0\}_{k\geqslant 0})}=\sum_{j=0}^{N-1}\left(\frac{1}{S_{j+1,j}}-1\right)+1. \tag{2.7}$$

Here $S_{j+1,j}$ is the transmission probability $j\to j+1$ in the interval $[j,j+1]$ in the absence of barriers, so that $S_{j+1,j}=\int_0^{+\infty}s_j(t)dt$. Then the total transmission probability at equilibrium is

$$S_{[0,N]}=\alpha_0^N S_{[0,N]}(\{0\}_{k\geqslant 0}|\{0\}_{k\geqslant 0}) \tag{2.8}$$

and everything behaves as in a Markovian case.

### D. Asymptotics for large $\lambda$

When $\lambda\to\infty$, each time the particle hits a barrier, it finds it in its equilibrium state and the system does not have any memory. For a given interval $[j,j+1]$, the effective transmission probability is then given by the equation

$$S_{j+1,j}^{(\mathrm{eff})}=S_{j+1,j}\alpha_0+S_{j+1,j}\alpha_1(1-S_{j+1,j}^{\mathrm{eff}})+(1-S_{j+1,j})\alpha_1 S_{j+1,j}^{(\mathrm{eff})},$$

which implies

$$S_{j+1,j}^{(\mathrm{eff})}=\frac{S_{j+1,j}}{1-\alpha_1(1-2S_{j+1,j})}. \tag{2.9}$$

Then we have the addition formula

$$\frac{1}{S_{[0,N]}}-1=\sum_{j=0}^{N-1}\left(\frac{1}{S_{j+1,j}^{(\mathrm{eff})}}-1\right). \tag{2.10}$$

When all the stochastic processes in $[j,j+1]$ are identical, we have

$$\frac{1}{S_{[0,N]}}-1=N\left(\frac{1-\alpha_1(1-2S)}{S}-1\right). \tag{2.11}$$

## III. ASYMPTOTIC IN THE LIMIT OF ALWAYS OPEN BARRIERS ($\alpha_0 \to 1$)

We study the asymptotic value of the transmission probability in the limit $\alpha_0 \to 1$, i.e., in the limit of all barriers open. Contrarily to the case of the limit $\alpha_0 \to 0$, we can use a perturbation expansion in terms of $\alpha_1 \equiv 1-\alpha_0$. Thus the method is completely different from the previous one and does not use the recursion relations of Appendix B. We shall assume that the barrier at site 0 is always open (to let in the particle) and we shall consider the total transmission probability $S_{[0,N]}$ (i.e., the probability that a particle entering at $t=0$ the array at point 0 leaves the interval $[0,N]$ through point $N$, the barriers being initially in their equilibrium state, except the barrier at 0, which is initially open). We look for an expansion in powers of $\alpha_1$,

$$S_{[0,N]}=S_{[0,N]}^{(0)}+S_{[0,N]}^{(1)}+\cdots, \tag{3.1}$$

where $S_{[0,N]}^{(k)}$ is the term in $\alpha_1^k$ and the term of order 0 and $S_{[0,N]}^{(0)}$ is exactly the transmission probability in the absence of barriers and is given by an ordinary addition law of the type

$$\frac{1}{S_{[0,N]}^{(0)}}-1=\sum_{j=0}^{N-1}\left(\frac{1}{S_j}-1\right), \tag{3.2}$$

where $S_j$ is the probability $\int_0^\infty s_j(t)dt$, i.e., the probability that the particle entering $[j,j+1]$ through point $j$ leaves this interval through point $j+1$ (in the absence of barriers). When all the intervals $[j,j+1]$ and the stochastic processes are identical, one recovers

$$S_{[0,N]}^{(0)}=\frac{S}{N-(N-1)S}, \quad S=\int_0^\infty s(t)dt. \tag{3.3}$$

The formula for $S_{[0,N]}^{(1)}$ is given by Eq. (C21) or (C22).

From these formulas, one finds immediately that

$$S_{[0,N]}^{(1)} \quad \text{increases with } \lambda, \tag{3.4}$$

$$\left.\frac{\partial S_{[0,N]}}{\partial \alpha_1}\right|_{\substack{\alpha_1=0\\ \lambda=0}}<0, \tag{3.5}$$

$$\left.\frac{\partial S_{[0,N]}}{\partial \alpha_1}\right|_{\substack{\alpha_1=0\\ \lambda=\infty}}=S_{[0,N]}^{(0)}[1-(N+1)S_{[0,N]}^{(0)}], \tag{3.6}$$

and so it is positive if and only if $S_{[0,N]}^{(0)}<1/(N+1)$. If

$$S_{[0,N]}^{(0)}<\frac{1}{N+1},$$

there exists a value $\lambda^*$ such that

$$\left.\frac{\partial S_{[0,N]}}{\partial \alpha_1}\right|_{\alpha_1=0}>0 \quad \text{for } \lambda>\lambda^*. \tag{3.7}$$

## IV. STOCHASTIC RESONANCE

In this section we prove that there is a stochastic resonance in terms of $\alpha_1$, namely, that the total transmission probability $S_{[0,N]}$ reaches a maximum, as a function of $\alpha_1$, for $\alpha_1 \neq 1$, at least for a certain interval of frequencies. We assume the following hypotheses.

(i) First we suppose that all the intervals $[j,j+1]$ and their stochastic processes inside them are identical. We denote by $S$ the transmission probability in a single interval $[j,j+1]$ (without a barrier).

(ii) Next we assume that $S<\frac{1}{2}$.

(iii) Then we assume that the particle starts at point 0 with a positive velocity, the state of the barrier at 0 is $\epsilon_0=0$ (open barrier), and all barriers $k$ ($k \geqslant 1$) are in the stationary state initially, that is, open with probability $\alpha_0$ and closed with probability $\alpha_1$.

Under these hypotheses, we can summarize the asymptotic results obtained in the previous sections. For (a) $\alpha_1=1$ we have from Eq. (2.1)

$$S_{[0,N]}|_{\alpha_1=1}=\frac{S_{10}(0,1)}{N-(N-1)S_{10}(0,1)}, \tag{4.1}$$

with $S_{10}(0,1)$ as in Eq. (2.1b),

$$S_{10}(0,1)=\frac{1}{2}\left(1-\hat{r}(\lambda)-\frac{\hat{s}(\lambda)^2}{1-\hat{r}(\lambda)}\right). \tag{4.2}$$

For (b) $\alpha_1=0$ we have from Eq. (3.3)

$$S_{[0,N]}|_{\alpha_1=0}=\frac{S}{N-(N-1)S}. \tag{4.3}$$

Hypothesis (ii) implies that

$$S_{[0,N]}|_{\alpha_1=0}<\frac{1}{N+1} \tag{4.4}$$

and from Eq. (3.6) we get

$$\left.\frac{\partial S_{[0,N]}}{\partial \alpha_1}\right|_{\alpha_1=0,\lambda=\infty}>0. \tag{4.5}$$

Moreover, from Eq. (3.5) we have

$$\left.\frac{\partial S_{[0,N]}}{\partial \alpha_1}\right|_{\alpha_1=0,\lambda=0}<0. \tag{4.6}$$

(c) For $\lambda \to 0$ we have

$$S_{[0,N]}|_{\alpha_1=1,\lambda=0}=0 \tag{4.7}$$

[see Eq. (2.8)] and

$$S_{[0,N]}|_{\alpha_1=0,\lambda=0}=\frac{S}{N-(N-1)S} \tag{4.8}$$

[see Eq. (2.7)]. (d) For $\lambda \to \infty$, we have from Eq. (2.11)

$$S_{[0,N]}=\frac{S^{(\text{eff})}}{N-(N-1)S^{(\text{eff})}}, \tag{4.9}$$

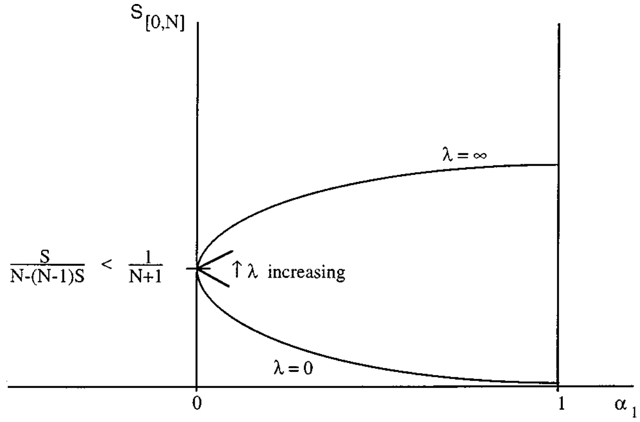


FIG. 2. Sketch of the graphs of $S_{[0,N]}$ as a function of $\alpha_1$ for various values of λ, including λ = 0 or ∞.

$$S^{(\mathrm{eff})}=\frac{S}{1-\alpha_1(1-2S)}, \tag{4.10}$$

$$S_{[0,N]} \text{ increases as a function of } \alpha_1 \text{ for } \lambda=\infty. \tag{4.11}$$

From these results alone we can sketch the graphs of $S_{[0,N]}$ as a function of $\alpha_1$ for $\lambda=0$ and $\infty$ (see Fig. 2): (i) For all λ, all graphs $S_{[0,N]}$ start at $\alpha_1=0$ at the point

$$\frac{S}{N-(N-1)S}<\frac{1}{N+1} \quad (S<\tfrac{1}{2}); \tag{4.12}$$

(ii) for $\lambda=0$, the graph of $S_{[0,N]}$ goes to 0 when $\alpha_1=1$ [Eq. (4.7)] and is decreasing as a function of $\alpha_1$ [Eq. (2.8)]; (iii) for $\lambda=\infty$, the graph of $S_{[0,N]}$ goes to $1/(N+1)$ and is an increasing function of $\alpha_1$ [Eq. (4.11)]; (iv) finally, $\partial S_{[0,N]}/\partial\alpha_1|_{\alpha_1=0}$ is an increasing function of λ [see Eq. (3.4)].

From the sketch of Fig. 2 it is clear that there must exist a certain interval of λ where $S_{[0,N]}$ has either a maximum or a minimum as a function of $\alpha_1$ because when λ varies from 0 to ∞, the graph varies from a decreasing to an increasing graph and surely for any value of λ, $S_{[0,N]}$ cannot be a constant function of $\alpha_1$. So there must exist certain intervals of λ so that the graph has a maximum or a minimum.

We are now in a position to prove the main result of this work: Under the assumptions of Eq. (4.1), in particular in the case of $S<\frac{1}{2}$, there exists an interval Λ of frequency so that when λ is in Λ, $S_{[0,N]}$ has a maximum as a function of $\alpha_1$ for a certain value of $\alpha_1$ between 0 and 1. Moreover, the interval Λ contains neither 0 nor ∞.

Using the sketches of the graphs (see Fig. 2), it is enough to prove that for certain frequencies λ,

$$S_{[0,N]}|_{\alpha_1=0}-S_{[0,N]}|_{\alpha_1=1}>0, \tag{4.13}$$

$$\left.\frac{\partial S_{[0,N]}}{\partial\alpha_1}\right|_{\alpha_1=0}>0. \tag{4.14}$$

Using Eq. (C21) in the form

$$\left.\frac{\partial S_{[0,N]}}{\partial\alpha_1}\right|_{\alpha_1=0}=\frac{S^{(0)}_{[0,N]}}{N-(N-1)S}\left[N-2S\sum_{k=1}^{N}\frac{1}{1-\hat{r}_{[0,k]}(\lambda)}\right],$$

we see that Eq. (4.14) is equivalent to

$$\frac{N}{S}>2\sum_{k=1}^{N}\frac{1}{1-\hat{r}_{[0,k]}(\lambda)}, \tag{4.15}$$

while, using Eqs. (4.1)–(4.3), we see that Eq. (4.13) is equivalent to

$$S-\frac{1}{2}[1-\hat{r}(\lambda)]+\frac{\hat{s}^2}{2-2(1-\hat{r})}>0. \tag{4.16}$$

We define the quantity

$$q=1-\hat{r}(\lambda) \tag{4.17}$$

so that Eq. (4.16) is equivalent to

$$\hat{s}^2>q^2-2qS\equiv g(q). \tag{4.18}$$

If $A$ is an upper bound of $\hat{r}_{[0,k]}$, it is sufficient to prove, instead of Eq. (4.15), that

$$A<1-2S. \tag{4.19}$$

In Appendix D 1, we show that we can replace Eq. (4.19) by the inequality

$$\frac{\hat{r}+\hat{s}^2(1-\hat{s}^{-2}\hat{r}^2)(1-2S)}{1-\hat{r}(1-2S)}<1-2S. \tag{4.20}$$

After a rearrangement, Eq. (4.20) can be rewritten

$$\frac{\hat{r}}{1-2S}+(\hat{s}^2-\hat{r}^2)<1-\hat{r}(1-2S),$$

which in terms of $q$ defined by Eq. (4.17) can be rewritten

$$\hat{s}^2<q^2+\frac{4S^2}{1-2S}q-\frac{4S^2}{1-2S}\equiv h(q). \tag{4.21}$$

Now, obviously

$$\hat{r}+\hat{s}\leqslant 1,$$

so that

$$S\leqslant q\leqslant 1, \quad \hat{s}^2\leqslant q^2, \quad \hat{s}^2\leqslant S^2. \tag{4.22}$$

The system of inequalities (4.18), (4.21), and (4.22) is discussed in Appendix D 2 and it is proved there that one can find an interval Λ of λ where these inequalities are simultaneously satisfied, which implies that Eqs. (4.13) and (4.14) are also both satisfied for λ in Λ so that the system presents a stochastic resonance in the sense that its transmission probability is maximized by a convenient choice of the parameter $\alpha_1$. Moreover, this result is valid for any $N$.

*Remark.* This result extends previous ones obtained for the case of $N=2$ barriers. However, for the method used in that case, $N$ cannot be extended to the case of general $N$. We notice why the simulation of [8] was not very conclusive: The reason is that the interval $\Lambda$ is rather small.

## CONCLUSION

The preceding conclusion is the main result of this article. In spite of the fact that the transmission probability cannot be calculated exactly, it was possible to prove analytically that, in relevant conditions, this transmission probability presents a stochastic resonance for a certain value of the average probability of the presence of the barriers (or average density of barriers) $\alpha_1$. The resonance is due to the interaction of two stochastic processes: the stochastic motion intrinsic to the particle and the fluctuations of the barriers. It should be pointed out that two characteristic times appear in this system: the relaxation time of the barriers and a characteristic return time, which is the average time needed by the particle to return to a barrier when all barriers are frozen. However, the resonance is not clearly related to the respective values of these times; in particular, the maximum of the transmission probability was obtained by varying $\alpha_1$ rather than the relaxation time of the barriers. Thus the comparison with other, more classical, cases of resonance is difficult.

The system studied here is a very special and idealized model of motion in a fluctuating environment of arbitrary size. Still this model has been used to describe the diffusion of a substrate in a protein as in [12], relying on previous experimental work and simulations [10,11]. The result in [12] is restricted to the case of two barriers and was treated using a Fokker-Planck–type approximation.

In subsequent work [13] we plan to study other models with two barriers where the barriers are correlated. We plan to prove there that there is a rich structure of the phase transition provided the correlation between the obstacle is strong enough.

## APPENDIX A: TRANSMISSION PROBABILITY FOR TWO BARRIERS

In this appendix we prove the various results stated in Sec. I.

### 1. Solution of the system of equations (1.4) and (1.5)

We rewrite this system of equations as follows. First we define two column vectors $X_a$, $a=0,1$, by

$$X_a=\begin{pmatrix} S_{a0}^{(\theta)} & (1,0) \\ S_{a0}^{(\theta)} & (1,1)\end{pmatrix}, \quad a=0,1. \tag{A1}$$

We know that

$$\sum_{\epsilon} \varphi_{\epsilon'\epsilon}\alpha_\epsilon=\alpha_{\epsilon'}, \tag{A2}$$

$$\varphi_{\epsilon 0}-\varphi_{\epsilon 1}=(-1)^\epsilon e^{-\lambda t}. \tag{A3}$$

After some lengthy computations we obtain

$$(1,-1)(X_0+X_1)=\frac{[s]^{\lambda+\theta}[1-(\alpha_0,\alpha_1)(X_0+X_1)]}{1-[r\varphi_{11}]^{\lambda+\theta}-\alpha_0[s]^{2\lambda+\theta}},$$
$$(1,-1)(X_1-X_0)=\frac{[s]^{\lambda+\theta}[1+(\alpha_0,\alpha_1)(X_1-X_0)]}{1-[r\varphi_{11}]^{\lambda+\theta}+\alpha_0[s]^{2\lambda+\theta}}, \tag{A4}$$

$$\begin{aligned}(\alpha_0,\alpha_1)(X_1+X_0)\Big\{&1-[r\varphi_{11}]^\theta-\alpha_1[s]^\theta \\ &-\frac{\alpha_1\alpha_0([s]^{\lambda+\theta})^2}{1-[r\varphi_{11}]^{\lambda+\theta}-\alpha_0[s]^{2\lambda+\theta}}\Big\} \\ =-&\frac{\alpha_1\alpha_0([s]^{\lambda+\theta})^2}{1-[r\varphi_{11}]^{\lambda+\theta}-\alpha_0[s]^{2\lambda+\theta}}+[(r+s)\varphi_{01}]^\theta \\ &+\alpha_0[s]^{\lambda+\theta},\end{aligned} \tag{A5}$$

$$\begin{aligned}(\alpha_0,\alpha_1)(X_1-X_0)\Big\{&1-[r\varphi_{11}]^\theta+\alpha_1[s]^\theta \\ &-\frac{\alpha_1\alpha_0([s]^{\lambda+\theta})^2}{1-[r\varphi_{11}]^{\lambda+\theta}+\alpha_0[s]^{2\lambda+\theta}}\Big\} \\ =&[(s-r)\varphi_{01}]^\theta+\alpha_0[s]^{\lambda+\theta} \\ &+\frac{\alpha_1\alpha_0([s]^{\lambda+\theta})^2}{1-[r\varphi_{11}]^{\lambda+\theta}+\alpha_0[s]^{2\lambda+\theta}}.\end{aligned} \tag{A6}$$

### 2. Asymptotic of $S_{a0}^{(\theta)}(\{\epsilon\})$ when $\alpha_0\to 0$, $\theta>0$

If we set $\alpha_0=0$ in Eqs. (A5) and (A6) we see that the second members tend to 0 [recall that $\varphi_{01}=\alpha_0(1-e^{-\lambda t})$], while the term in curly brackets of these equations (A5) and (A6) stays away from 0 because $\theta>0$ [it is $1-\hat{r}(\theta)\pm\hat{s}(\theta)$], so that the matrix product $(0,1)X_a$ vanishes for $a=0,1$. Thus

$$S_{10}^{(\theta)}(1,1)=S_{00}^{(\theta)}(1,1)=0 \quad (\alpha_0=0). \tag{A7}$$

Then Eqs. (A3) and (A4) give

$$S_{00}^{(\theta)}(1,0)=0, \tag{A8}$$

$$S_{10}^{(\theta)}(1,0)=\frac{\hat{s}(\lambda+\theta)}{1-\hat{r}(\lambda+\theta)}. \tag{A9}$$

Using Eqs. (1.2) and (1.3), it is easy to obtain $S_{10}^{(\theta)}(0,\epsilon)$ by setting $\alpha_0=0$ in these equations and using the previously obtained limiting values

$$S_{00}^{(\theta)}(0,0)=\hat{r}(\theta+\lambda)+\frac{[\hat{s}(\lambda+\theta)-\hat{s}(2\lambda+\theta)]\hat{s}(\lambda+\theta)}{1-\hat{r}(\lambda+\theta)},$$
$$S_{00}^{(\theta)}(0,1)=\hat{r}(\theta+\lambda)+\frac{\hat{s}(\lambda+\theta)^2}{1-\hat{r}(\lambda+\theta)},$$
$$S_{10}^{(\theta)}(0,0)=\hat{s}(\lambda+\theta)+\frac{[\hat{r}(\lambda+\theta)-\hat{r}(2\lambda+\theta)]\hat{s}(\lambda+\theta)}{1-\hat{r}(\lambda+\theta)},$$
$$S_{10}^{(\theta)}(0,1)=0. \tag{A10}$$

#### 3. Asymptotic of $S_{a0}^{(\theta)}(\{\epsilon\})$ when $\alpha_0 \to 0$, $\theta=0$

Now we set $\theta=0$ and consider Eqs. (A5) and (A6) when $\alpha_0 \to 0$. The second member is proportional to $\alpha_0$. The term in curly brackets in Eq. (A5) is also proportional to $\alpha_0$, while the term in curly brackets in Eq. (A6) stays away from 0. A detailed analysis proves that the matrix products (0,1) $\times(X_0-X_1)$ and $(0,1)(X_0+X_1)$ vanish when $\alpha_0 \to 0$. From this we deduce that

$$S_{00}(1,1)=S_{10}(1,1)=\tfrac{1}{2} \tag{A11}$$

(using the conservation of probability to leave the interval [0,1] and the symmetry with respect to the exchange of 0 and 1).

Using Eqs. (A3) and (A4), it is easy to obtain when $\alpha_0 \to 0$

$$S_{00}(1,0)=\frac{1}{2}-\frac{1}{2}\frac{\hat{s}(\lambda)}{1-\hat{r}(\lambda)} \quad (\alpha_0 \to 0). \tag{A12}$$

Using Eqs. (1.6) and (1.7), one can also compute the various limits $S_{a0}(0,\epsilon)$ when $\alpha_0 \to 0$. Recalling that by conservations of probabilities

$$S_{00}(\{\epsilon\})+S_{10}(\{\epsilon\})=1, \tag{A13}$$

we obtain from Eqs. (1.2)–(1.7), (A12), and (A13)

$$S_{00}(0,0)=\frac{1}{2}\left[1+\hat{r}(\lambda)-\hat{s}(\lambda)+\frac{\hat{s}(\lambda)}{1-\hat{r}(\lambda)}[\hat{s}(\lambda)-\hat{r}(\lambda)\right.$$
$$\left.-\hat{s}(2\lambda)+\hat{r}(2\lambda)]\right], \tag{A14}$$

$$S_{00}(0,1)=\frac{1}{2}\left[1+\hat{r}(\lambda)+\frac{\hat{s}(\lambda)^2}{1-\hat{r}(\lambda)}\right]. \tag{A15}$$

#### 4. Detailed transmission probabilities

Here we determine for future use the asymptotic values of $S_{a0}(\{\epsilon'\}|\{\epsilon\})$. We start first with the system for $S_{a0}(0,0|1,\epsilon)$. It is easy to verify

$$S_{00}(0,0|1,1)=[r\varphi_{01}^2]+[r\varphi_{11}\varphi_{\epsilon 1}]S_{00}(0,0|1,\epsilon)$$
$$+[s\varphi_{11}\varphi_{\epsilon 1}]S_{10}(0,0|1,\epsilon), \tag{A16}$$

$$S_{00}(0,0|1,0)=[r\varphi_{00}\varphi_{01}]+[r\varphi_{\epsilon 0}\varphi_{11}]S_{00}(0,0|1,\epsilon)$$
$$+[s\varphi_{10}\varphi_{\epsilon 1}]S_{10}(0,0|1,\epsilon), \tag{A17}$$

$$S_{10}(0,0|1,1)=[s\varphi_{01}^2]+[s\varphi_{\epsilon 1}\varphi_{11}]S_{00}(0,0|1,\epsilon)$$
$$+[r\varphi_{\epsilon 1}\varphi_{11}]S_{10}(0,0|1,\epsilon), \tag{A18}$$

$$S_{10}(0,0|1,0)=[s\varphi_{00}\varphi_{01}]+[s\varphi_{10}\varphi_{\epsilon 1}]S_{00}(0,0|1,\epsilon)$$
$$+[r\varphi_{11}\varphi_{\epsilon 0}]S_{10}(0,0|1,\epsilon). \tag{A19}$$

If we set $\alpha_0=0$ in Eqs. (A18) and (A17) we obtain

$$S_{00}(0,0|1,1)=S_{10}(0,0|1,1), \tag{A20}$$

$$S_{00}(0,0|1,0)=\frac{1-\hat{r}(\lambda)-\hat{s}(\lambda)}{1-\hat{r}(\lambda)}S_{00}(0,0|1,1). \tag{A21}$$

Setting $\alpha_0=0$ in Eq. (A15), we get, using Eq. (A20),

$$S_{10}(0,0|1,0)=\frac{1-\hat{r}(\lambda)-\hat{s}(\lambda)}{1-\hat{r}(\lambda)}S_{00}(0,0|1,1). \tag{A22}$$

Now, when we set $\alpha_0=0$ in Eq. (A15), this equation degenerates. So we use Eqs. (A20)–(A22) in Eq. (A16) to obtain

$$S_{00}(0,0|1,1)\left\{1-[r\varphi_{11}\varphi_{01}]\frac{1-\hat{r}(\lambda)-\hat{s}(\lambda)}{1-\hat{r}(\lambda)}\right.$$
$$\left.-[s\varphi_{11}\varphi_{01}]\frac{1-\hat{r}(\lambda)-\hat{s}(\lambda)}{1-\hat{r}(\lambda)}-[r\varphi_{11}^2]\right\}$$
$$-[s\varphi_{11}^2]S_{10}(0,0|1,1)=0(\alpha_0^2) \tag{A23}$$

and Eq. (A18) gives

$$S_{10}(0,0|1,1)\{1-[r\varphi_{11}^2]\}$$
$$-S_{00}(0,0|1,1)\left\{[s\varphi_{11}^2]+[s\varphi_{01}\varphi_{11}]\frac{1-\hat{r}(\lambda)-\hat{s}(\lambda)}{1-\hat{r}(\lambda)}\right.$$
$$\left.+[r\varphi_{01}\varphi_{11}]\frac{1-\hat{r}(\lambda)-\hat{s}(\lambda)}{1-\hat{r}(\lambda)}\right\}=0(\alpha_0^2). \tag{A24}$$

Now the determinant of the $2\times 2$ linear system (A23) and (A24) is

$$\alpha_0 2S[1-\hat{r}(\lambda)-\hat{s}(\lambda)]\left(2-\frac{1-\hat{r}(\lambda)-\hat{s}(\lambda)}{1-\hat{r}(\lambda)}\right).$$

It is exactly of order $\alpha_0$, while the second members of Eqs. (A23) and (A24) are of order $\alpha_0^2$. As a consequence,

$$S_{00}(0,0|1,1)=S_{10}(0,0|1,1)=0. \tag{A25}$$

From Eqs. (A21) and (A22)

$$S_{00}(0,0|1,0)=S_{10}(0,0|1,0)=0. \tag{A26}$$

Then for $\alpha_0=0$,

$$S_{10}(0,0|0,1)=[r\varphi_{10}]S_{00}(0,0|1,1)+[s\varphi_{\epsilon 0}]S_{10}(0,0|1,\epsilon),$$

so that

$$S_{10}(0,0|0,1)=S_{00}(0,0|0,1)=0. \tag{A27}$$

From these results one finds

$$S_{00}(0,0|0,0)=\hat{r}(2\lambda). \tag{A28}$$

Finally, all the other transmission probabilities can be computed easily.

## APPENDIX B: TRANSMISSION PROBABILITIES FOR *N* BARRIERS

We are now interested in the following quantities. We call

$$R_{[0,N]}(t;\{\epsilon'\}|\{\epsilon\})dt$$

the probability that the particle, starting from 0 at time $t=0$, the states of the barriers being $\{\epsilon_k\}_{k=0,\ldots,N}\equiv\{\epsilon\}$, leaves the interval $[0,N]$ for the first time at point 0 between times $t$ and $t+dt$, the states of the barriers at that time being $\{\epsilon'_k\}_{k=0,\ldots,N}\equiv\{\epsilon'\}$ (with of course $\epsilon'_0=0$). In the same manner, we shall also need

$$R_{[0,N]}(t;\{\epsilon'\}|1^-,\{\epsilon\})dt,$$

which is defined exactly in the same way, except that now the particle starts from 1 with a negative velocity (i.e., from 1 in the first interval [0,1] instead of 0). Finally,

$$R_{[1,N]}(t;\{\epsilon'\}|\{\epsilon\})dt$$

is defined as the probability that the particle starting at $t=0$ from 1 (with positive velocity), the states of the barriers being $\{\epsilon_k\}_{k=1,\ldots,N}=\{\epsilon\}$ in the interval $[1,N]$, leaves $[1,N]$ for the first time through 1, between times $t$ and $t+dt$, the states of the barriers being $\{\epsilon'_k\}_{k=1,\ldots,N}=\{\epsilon'\}$ (with $\epsilon'_1=0$).

We shall also need the Laplace transforms

$$R^{\theta}_{[0,N]}(\{\epsilon'\}|\{\epsilon\})=\int_0^{\infty}e^{-\theta t}R_{[0,N]}(t;\{\epsilon'\}|\{\epsilon\})dt.$$

When $\theta=0$ we skip the index $\theta=0$,

$$R_{[0,N]}(\{\epsilon'\}|\{\epsilon\})=R^{\theta}_{[0,N]}(\{\epsilon'\}|\{\epsilon\})|_{\theta=0}.$$

We also denote the aggregate quantities

$$R_{[0,N]}(t|\{\epsilon\})=\sum_{\{\epsilon'\}}R_{[0,N]}(t;\{\epsilon'\}|\{\epsilon\}),$$

$$R^{(\theta)}_{[0,N]}(\{\epsilon\})=\sum_{\{\epsilon'\}}R^{(\theta)}_{[0,N]}(\{\epsilon'\}|\{\epsilon\}).$$

### 1. Recursion relations

We write down recurrence relations for the matrices $R^{\theta}_{[0,N]}$. Namely, we assume that

$$\epsilon'_0=0. \tag{B1}$$

Summing up the different contributions, as done in Eqs. (1.4) and (1.5), we find

$$\begin{aligned}R^{\theta}_{[0,N]}&(\{\epsilon'\}|\{\epsilon\})\\&=\left[S_{00}(t;0,\epsilon'_1|\epsilon_0,\epsilon_1)\prod_{k\geqslant 2}\varphi_{\epsilon'_k\epsilon_k}\right]^{(\theta)}\\&\quad+R^{\theta}_{[0,N]}(\{\epsilon'\}|1^-;\eta'_0,0,\{\eta'_k\}_{k\geqslant 2})\\&\quad\times[R_{[1,N]}(t;0_1,\{\eta'_k\}_{k\geqslant 2}|0_1,\{\eta_k\})\varphi_{\eta'_0\eta_0}]^{(\theta)}\\&\quad\times\left[S_{10}(t;\eta_0,0|\epsilon_0,\epsilon_1)\prod_{k\geqslant 2}\varphi_{\eta_k\epsilon_k}\right]^{(\theta)}\end{aligned}\tag{B2}$$

and we have

$$\begin{aligned}R^{\theta}_{[0,N]}&(\{\epsilon'\}|1^-;\epsilon_0,0_1,\{\epsilon_k\}_{k\geqslant 2})\\&=\left[S_{10}(t;\epsilon'_1,0|0,\epsilon_0)\prod_{k\geqslant 2}\varphi_{\epsilon'_k\epsilon_k}\right]^{(\theta)}\\&\quad+R^{\theta}_{[0,N]}(\{\epsilon'\}|1^-;\eta'_0,0,\{\eta'_k\}_{k\geqslant 2})\\&\quad\times[R_{[1,N]}(t;0_1,\{\eta'_k\}_{k\geqslant 2}|0,\{\eta_k\}_{k\geqslant 2})\varphi_{\eta'_0\eta_0}]^{\theta}\\&\quad\times\left[S_{00}(t;0,\eta_0|0,\epsilon_0)\prod_{k\geqslant 2}\varphi_{\eta_k\epsilon_k}\right]^{\theta}.\end{aligned}\tag{B3}$$

Although this system of equations (B2) and (B3) seems prohibitively complicated, we shall show that it reduces to extremely simple forms in certain limiting cases, namely, $\alpha_0\to 0$, $\alpha_0\to 1$, $\lambda\to 0$, and $\lambda\to\infty$.

### 2. Asymptotic for $\alpha_0\to 0$

#### *a. A Laplace transform result for θ>0*

We first prove the following preliminary result. For $\theta>0$ and $\alpha_0=0$ we have

$$R^{\theta}_{[0,N]}(\{\epsilon'\}|0,1,\{\epsilon_k\}_{k\geqslant 2})=\left[S_{00}(t;0,\epsilon'_1|0,1)\prod_{k\geqslant 2}\varphi_{\epsilon'_k\epsilon_k}\right]^{(\theta)}.\tag{B4}$$

To show this, we consider Eq. (B2) and we prove that

$$\left[S_{10}(t;\eta_0,0|0,1)\prod_{k\geqslant 2}\varphi_{\eta_k\epsilon_k}\right]^{(\theta)}=0\quad\text{for }\theta>0,\ \alpha_0=0.\tag{B5}$$

This results from the expansion of $\Pi_{k\geqslant 2}\varphi_{\eta_k\epsilon_k}$, so that the quantity (B5) is a sum of Laplace transforms (at positive value of the Laplace parameter because $\theta>0$) of $S_{10}(t;\eta_0,0|0,1)$ and this is 0 by Eq. (A14).

#### *b. The case θ>0 with all barriers closed (except the barrier at 0)*

Using the results (B4) and Eqs. (A12) and (A13), we can prove that for $\theta=0$, $\{\epsilon_k\}=\{0,\{1\}_{k\geqslant 1}\}$, and $\alpha_0\to 0$, we have

$$R_{[0,N]}(\{\epsilon'\}|0,\{1\}_{k\geqslant 1})=R_{[0,N]}(\{\epsilon'\}|0,\{1\}_{k\geqslant 1})\,\delta_{\epsilon'_0 0}\prod_{k\geqslant 1}\delta_{\epsilon'_k 1},\tag{B6}$$

$$R_{[0,N]}(\{\epsilon'\}|1^-;\epsilon_0,0,\{1\}_{k\geq 2})$$
$$=R_{[0,N]}(\{\epsilon'\}|1^-;\epsilon_0,0,\{1\}_{k\geq 2})\,\delta_{\epsilon'_0 0}\prod_{k\geq 1}\delta_{\epsilon'_k}. \tag{B7}$$

The proof is by recursion on $N$, the case $N=1$ being proved in Sec. I. Finally, Eqs. (B2) and (B3) become

$$\begin{aligned}R_{[0,N]}(\{\epsilon'\}|0,\{1\}_{k\geq 1}) &= S_{00}(0,1|0,1)\,\delta_{\epsilon'_0 0}\prod_{k\geq 1}\delta_{\epsilon'_k 1}\\ &\quad + R_{[0,N]}(\{\epsilon'\}|1^-;1,0,\{1\}_{k\geq 2})\\ &\quad \times R_{[1,N]}(0_1,\{1\}_{k\geq 2}|0_1,\{1\}_{k\geq 2})\\ &\quad \times S_{10}(1,0|0,1),\end{aligned} \tag{B8}$$

$$\begin{aligned}R_{[0,N]}(\{\epsilon'\}|1^-;1,0,\{1\}_{k\geq 2}) &= S_{10}(1,0|0,1)\prod_{k\geq 1}\delta_{\epsilon'_k 1}\delta_{\epsilon'_0 0}\\ &\quad + R_{[0,N]}(\{\epsilon'\}|1^-;1,0,\{1\}_{k\geq 2})\\ &\quad \times R_{[1,N]}(0,\{1\}_{k\geq 2}|0,\{1\}_{k\geq 2})S_{00}(0,1|0,1).\end{aligned} \tag{B9}$$

#### *c. Solution of the system (B8) and (B9)*

First we notice, using the notation of Sec. I,

$$S_{00}(0,1|0,1)=S_{00}(0,1),$$

$$S_{10}(1,0|0,1)=S_{10}(0,1),$$

so that

$$R_{[0,N]}(0,\{1\}_{k\geq 1}|0,\{1\}_{k\geq 1})=S_{00}(0,1)+\frac{S_{10}(0,1)^2R_{[1,N]}(0,\{1\}_{k\geq 2}|0,\{1\}_{k\geq 2})}{1-S_{00}(0,1)R_{[1,N]}(0,\{1\}_{k\geq 2}|0,\{1\}_{k\geq 2})}. \tag{B10}$$

Notice that

$$R_{[0,N]}(0,\{1\}_{k\geq 1}|0,\{1\}_{k\geq 1})\equiv R_{[0,N]}(0,\{1\}_{k\geq 1})$$

is the total return probability (whatever the final states of the barriers are) and the total transmission probability is thus

$$S_{[0,N]}(0,\{1\}_{k\geq 1})=1-R_{[0,N]}(0,\{1\}_{k\geq 1}). \tag{B11}$$

A little manipulation gives

$$\frac{1}{S_{[0,N]}(0,\{1\}_{k\geq 1})}=\frac{1}{S_{[1,N]}(0,\{1\}_{k\geq 2})}+\frac{1}{S_{10}(0,1)}-1 \tag{B12}$$

which leads to the addition formula (2.1).

## APPENDIX C: DERIVATION OF THE ASYMPTOTICS FOR $\alpha_1\to 0$

We now derive the first correction for small $\alpha_1$, denoted by $S^{(1)}_{[0,N]}$ in Eq. (3.1). We consider the set of all trajectories starting from 0 at time 0, leaving $[0,N]$ at a certain time, through $N$ and the total weight of these trajectories, which is $S_{[0,N]}$. We can write the matrix $\varphi_{\epsilon'\epsilon}(t)$ as

$$\boldsymbol{\varphi}(t)=\begin{pmatrix}1 & 1-e^{-\lambda t}\\ 0 & e^{-\lambda t}\end{pmatrix}+\alpha_1(1-e^{-\lambda t})\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\equiv\boldsymbol{\varphi}^{(0)}+\boldsymbol{\varphi}^{(1)}. \tag{C1}$$

and the stationary state

$$\boldsymbol{\alpha}=\begin{pmatrix}1\\0\end{pmatrix}+\alpha_1\begin{pmatrix}-1\\1\end{pmatrix}=\boldsymbol{\alpha}^{(0)}+\boldsymbol{\alpha}^{(1)}. \tag{C2}$$

A trajectory contributing to $S_{[0,N]}$ is determined by the data

$$[p\geq N,(k_0,\epsilon_0,t_0),(k_1,\epsilon_1,t_1),\ldots,(k_p,\epsilon_p,t_p)], \tag{C3}$$

where $p$ is an integer greater than or equal to $N$, $k_0=0$, $k_1,\ldots,k_p=N$ are the positions of the successive barriers visited by the particle, $\epsilon_0=0$, $\epsilon_1,\ldots,\epsilon_p$ are the states of the barrier when the particle visits them, and $t_0=0\leq t_1\leq\cdots\leq t_p$ are the instants of visit of the barriers. The weight of a trajectory (C3) contains various factors of the type $r_l(t_j-t_{j-1})$ or $s_l(t_i-t_{i-1})$, which are independent of $\alpha_1$, and also contains factors depending on $\alpha_1$, which are of two types for each $(k_l,\epsilon_l,t_l)$, $l\geq 1$. One has (i) a factor $\alpha_{\epsilon_l}$ if the barrier is visited for the first time and thus found in its stationary state with probability $\alpha_{\epsilon_l}$ and (ii) a factor $\varphi_{\epsilon_l\epsilon_j}(t_l-t_j)$ if the barrier is not visited for the first time and if $(k_j,\epsilon_j,t_j)$ was the event of a visit of that same barrier before $(k_l,\epsilon_l,t_j)$.

To recover $S_{[0,N]}$, one sums over $p\geq N$ and over all possible $(k_j,\epsilon_j)$ and integrates over $t_0\leq t_1\leq\cdots\leq t_p<+\infty$. Obviously, $S^{(0)}_{[0,N]}$ is obtained for $\alpha_0=1$, namely, by replacing all the $\alpha_{\epsilon_l}$ and $\varphi_{\epsilon_l\epsilon_j}$ by 1 [as it should be from Eqs. (C1) and (C2)].

To obtain $S^{(1)}_{[0,N]}$, one must consider the correction due to a single $\alpha_1$, in the weight of all possible trajectories. These corrections can come only from factor (i) or (ii) above. In the corresponding factor, we shall use $\alpha^{(1)}_{\epsilon_l}$ or $\varphi^{(1)}_{\epsilon_l\epsilon_j}(t_l-t_j)$ and in other factors one shall use $\alpha^{(0)}_{\epsilon}$ or $\varphi^{(0)}_{\epsilon\epsilon'}(t-t')$ [as in Eqs. (C1) and (C2)].

#### 1. Corrections coming from an event of a first visit $(k_l,\epsilon_l,t_l)$

There are two types of such corrections according to the value of $\epsilon_l$.

*(a) $\epsilon_l=0$ (i.e., the barrier is open).* For the term $\epsilon_l$, $\alpha^{(1)}_0=-\alpha_1$. In the correction, all other barriers visited for

the first time are treated as open; for multiple visits, they pick up the factor $\varphi_{00}^{(0)}=1$.

If $k_l=k$, the contribution is

$$-\alpha_1 S_{[0,k]}^{(0)}\sum_{q=0}^{\infty}(R_{[k,N]}^{(0)}R_{[k,0]}^{(0)})^q S_{[k,N]}^{(0)},$$

where $R_{[a,b]}^{(0)}$ is the probability that the particle entering the interval $[a,b]$ through $a$ leaves it through $b$ (without a barrier in the interval $[a,b]$). The total contribution is $C_{a1}$, obtained by summing over $k$,

$$C_{a1}=-\alpha_1\sum_{k\geqslant 1}S_{[0,k]}^{(0)}(1-R_{[k,N]}^{(0)}R_{[k,0]}^{(0)})^{-1}S_{[k,N]}^{(0)}.$$

However, it is also clear for any $k$ that

$$S_{[0,N]}^{(0)}=S_{[0,k]}^{(0)}\sum_{q=0}^{\infty}(R_{[k,N]}^{(0)}R_{[k,0]}^{(0)})^q S_{[k,N]}^{(0)}, \tag{C4}$$

so that

$$C_{11}=-N\alpha_1 S_{[0,N]}^{(0)}. \tag{C5}$$

*(b)* $\epsilon_l=1$ *(the barrier is closed).* In this case the term in $\epsilon_l$ gives $\alpha_1$ as a contribution. The particle is reflected and has to return at least once to the barrier $k_l$. At the first return in $k_l$, the correction must be calculated using $\varphi_{\epsilon_j\epsilon_l}^{(0)}$ or

$$\varphi_{\epsilon_j 1}^{(0)}(t)=\begin{cases}1-e^{-\lambda t} & \text{if}\quad \epsilon_j=0\\ e^{-\lambda t} & \text{if}\quad \epsilon_j=1.\end{cases} \tag{C6}$$

Once the barrier at $k_l$ has been found open, the subsequent $\varphi^{(0)}$ are all 1 for all subsequent returns to $k_l$.

If the barrier at $k_l$ is still closed, the particle must visit this barrier a second time, giving a factor of the type (C5). After integration, we find a correction $C_b$, which is

$$C_{(\mathrm{b})}=\alpha_1\sum_{k\geqslant 1}S_{[0,k]}^{(0)}\left(\sum_{q=0}^{\infty}(R_{[k,0]}^{(0)}-\hat r_{[k,0]}^{(0)}(\lambda))(\hat r_{[k,0]}^{(0)}(\lambda))^q\right)\times(1-R_{[k,0]}^{(0)}R_{[k,N]}^{(0)})^{-1}S_{[k,N]}^{(0)}. \tag{C7}$$

Here $r_{[a,b]}^{(0)}(t)dt$ is the probability that the particle entering $[a,b]$ through $a$ leaves $[a,b]$ through $a$ between $t$ and $t+dt$ without any barrier and

$$\hat r_{[a,b]}^{(0)}(\lambda)=\int_0^{\infty}e^{-\lambda t}r_{[a,b]}^{(0)}(t)dt,$$

$$R_{[a,b]}^{(0)}\equiv \hat r_{[a,b]}(0).$$

Using Eq. (C4), one can rewrite Eq. (C7) as

$$C_{(\mathrm{b})}=\alpha_1 S_{[0,N]}^{(0)}\sum_{k\geqslant 1}(R_{[k,0]}^{(0)}-\hat r_{[k,0]}^{(0)}(\lambda))(1-\hat r_{[k,0]}^{(0)}(\lambda))^{-1}.$$

Because $R_{[k,0]}^{(0)}=1-S_{[k,0]}^{(0)}$ one has

$$C_{(\mathrm{b})}=N\alpha_1 S_{[0,N]}^{(0)}-\alpha_1\sum_{k=1}^{N}S_{[k,0]}^{(0)}(1-\hat r_{[k,0]}^{(0)}(\lambda))^{-1}. \tag{C8}$$

**2. Corrections due to an event of a second visit**

In this case, one uses for the event $(k_l,\epsilon_l,t_l)$ the $\varphi^{(1)}$ term and for all other events we use $\alpha_\epsilon^{(0)}=\delta_{\epsilon 0}$ for the first visit and $\varphi=1$ for all successive visits, until a certain $(k_l,\epsilon_l,t_l)$ for which one uses $\varphi_{\epsilon_l 0}^{(1)}(t_l-t_j)$ (because the previous $\epsilon_j$ was open). Thereafter, one continues to use $\varphi^{(0)}$ for all subsequent visits. We have two cases.

(i) At the last passage $t_j$ at $k_j=k_l$, before $t_l$, the velocity was positive. The part of the trajectory from 0 to $t_j$ will give a contribution, after integration over time

$$\sum_{q\geqslant 0}S_{[0,k]}^{(0)}(R_{[k,0]}^{(0)}R_{[k,N]}^{(0)})^q=S_{[0,k]}^{(0)}(1-R_{[k,0]}^{(0)}R_{[k,N]}^{(0)})^{-1}. \tag{C9}$$

Then comes the term $\varphi_{\epsilon_l 0}^{(1)}(t_l-t_j)$, which is $-(-1)^{\delta_{\epsilon_l 0}}\alpha_1(1-e^{\lambda(t_l-t_j)})$, which after integration over $t_l$ will give the factor

$$-(-1)^{\delta_{\epsilon_l 0}}\alpha_1(R_{[k,N]}^{(0)}-\hat r_{[k,N]}^{(0)}(\lambda)), \tag{C10}$$

corresponding to the loop from $k_j=k_l$ back to $k_l$ inside $[k_l,N]$.

(a) If at $t_l$ the barrier is closed, the particle can perform an arbitrary number of returns from $k_l$ to $k_l$ within $[k_l,N]$, finding the barrier closed and eventually leaving $[k_l,N]$ through $N$, which has a probability

$$S_{[k,N]}^{(0)}(1-\hat r_{[k,N]}(\lambda))^{-1}, \tag{C11}$$

or it can perform an arbitrary number of returns from $k_l$ to $k_l$ within $[k_l,N]$, finding the barrier closed, then it returns to $k_l$, finding the barrier open, and then moves in the whole system with all barriers open until it leaves through $N$. The probability is

$$S_{[k,N]}^{(0)}(1-R_{[k,0]}^{(0)}R_{[k,N]}^{(0)})^{-1}R_{[k,0]}^{(0)}(R_{[k,N]}^{(0)}-\hat r_{[k,N]}(\lambda))\times(1-\hat r_{[k,N]}(\lambda))^{-1}. \tag{C12}$$

So this case (i a) gives a contribution $C_{(\mathrm{i\,a})}$ that is the product of Eqs. (C9) and (C10) and the sum of Eqs. (C11) and (C12), that is,

$$C_{(\mathrm{i\,a})}=\alpha_1\sum_{k\geqslant 1}S_{[0,k]}^{(0)}(1-R_{[k,0]}^{0}R_{[k,N]}^{(0)})^{-1}\times[R_{[k,N]}^{(0)}-\hat r_{[k,N]}(\lambda)]\times S_{[k,N]}^{(0)}[1-\hat r_{[k,N]}(\lambda)]^{-1}\times\left[1+\frac{R_{[k,0]}^{(0)}[R_{[k,N]}^{(0)}-\hat r_{[k,N]}(\lambda)]}{1-R_{[k,0]}^{(0)}R_{[k,N]}^{(0)}}\right].$$

Using again Eq. (C4) this is

$$C_{(\mathrm{i\,a})}=\alpha_1 S_{[0,N]}^{(0)}\sum_{k\geqslant 1}\left(\frac{1-R_{[k,0]}^{(0)}\hat r_{[k,N]}(\lambda)}{1-R_{[k,0]}^{(0)}R_{[k,N]}^{(0)}}\right)\times\left(\frac{R_{[k,N]}^{(0)}-\hat r_{[k,N]}(\lambda)}{1-\hat r_{[k,N]}(\lambda)}\right). \tag{C13}$$

(b) If at time $t_l$ the barrier is open ($\epsilon_l=0$) it remains open all the time (for the first-order contributions). This gives the contribution

$$C_{(\mathrm{i\,b})}=-\alpha_1\sum_k \frac{S^{(0)}_{[k,N]}R^{(0)}_{[k,0]}}{1-R^{(0)}_{[k,0]}R^{(0)}_{[k,N]}}[R^{(0)}_{[k,N]}-\hat{r}_{[k,N]}(\lambda)] \times\frac{S^{(0)}_{[0,k]}}{1-R^{(0)}_{[k,0]}R^{(0)}_{[k,N]}} \tag{C14}$$

or

$$C_{(\mathrm{i\,b})}=-\alpha_1 S^{(0)}_{[0,N]}\sum_{k\geqslant 1} \frac{R^{(0)}_{[k,0]}}{1-R^{(0)}_{[k,0]}R^{(0)}_{[k,N]}}[R^{(0)}_{[k,N]}-\hat{r}_{[k,N]}(\lambda)].$$

Summing Eqs. (C13) and (C14) gives the total contribution of case (i)

$$C_{(\mathrm{i})}=\alpha_1 S^{(0)}_{[0,N]}\sum_{k\geqslant 1}^{N-1} S^{(0)}_{[k,0]}\frac{R^{(0)}_{[k,N]}-\hat{r}_{[k,N]}(\lambda)}{(1-R_{[k,0]}R_{[k,N]})[1-\hat{r}_{[k,N]}(\lambda)]}. \tag{C15}$$

(ii) At the last passage $t_j$ at $k_j=k_l$ before $t_l$, the velocity was negative. The part of the trajectory from 0 to $t_j$ will give after integration over $t_j$

$$R^{(0)}_{[k,N]}(1-R^{(0)}_{[k,0]}R^{(0)}_{[k,N]})^{-1}S^{(0)}_{[0,k]}. \tag{C16}$$

The loop from $k_j=k_l$ at $t_j$ to $k_l$ at time $t_l$ gives the $\alpha_1$ contribution (after integration on $t_l$)

$$\mp(-1)^{\delta_{\epsilon_l 0}}[R_{[k,0]}-\hat{r}_{[k,0]}(\lambda)]. \tag{C17}$$

(a) If $\epsilon_l=1$ at time $t_l$, the trajectory can perform an arbitrary number of returns from $k_l$ to $k_l$ in $[0,k]$ finding the barrier closed before the first time when it finds it open and leaving to $N$. This gives, using Eqs. (C16) and (C17),

$$C_{(\mathrm{ii\,a})}=\alpha_1\frac{R^{(0)}_{[k,N]}S^{(0)}_{[0,k]}}{1-R^{(0)}_{[k,0]}R^{(0)}_{[k,N]}}\frac{[R^{(0)}_{[k,0]}-\hat{r}_{[k,0]}(\lambda)]}{1-\hat{r}_{[k,0]}(\lambda)} \times\frac{S_{[k,N]}}{1-R^{(0)}_{[k,0]}R^{(0)}_{[k,N]}}. \tag{C18}$$

(b) If $\epsilon_l=0$, the barrier is open. It will remain open for the calculation of the corrections. The contribution of trajectories from $k$ to $N$ is

$$S^{(0)}_{[k,N]}\frac{1}{1-R^{(0)}_{[k,0]}R^{(0)}_{[k,N]}},$$

which together with Eq. (C16) and (C17) gives

$$C_{(\mathrm{ii\,b})}=-\alpha_1\frac{S^{(0)}_{[k,N]}}{1-R^{(0)}_{[k,0]}R^{(0)}_{[k,N]}}[R_{[k,0]}-\hat{r}_{[k,0]}(\lambda)] \times\frac{R^{(0)}_{[k,N]}S^{(0)}_{[0,k]}}{1-R^{(0)}_{[k,0]}R^{(0)}_{[k,N]}}. \tag{C19}$$

The total contribution $C_{(\mathrm{ii})}$ of the case (ii) is given by summing Eqs. (C18) and (C19):

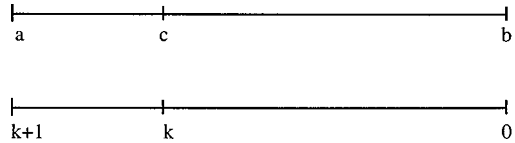


FIG. 3. Intervals $[a,c]$ and $[c,b]$ introduced in Appendix C.

$$C_{(\mathrm{ii})}=-\alpha_1 S^{(0)}_{[0,N]}\sum_{k\geqslant 0} S^{(0)}_{[k,0]}\left(\frac{R^{(0)}_{[k,0]}-\hat{r}_{[k,0]}}{1-\hat{r}_{[k,0]}}\right)\left(\frac{R^{(0)}_{[k,N]}}{1-R^{(0)}_{[k,0]}R^{(0)}_{[k,N]}}\right). \tag{C20}$$

### 3. Addition of all corrections

If one sums the various contributions $C_{(\mathrm{a})},C_{(\mathrm{b})},C_{(\mathrm{i})},C_{(\mathrm{ii})}$, one obtains the corrections

$$S^{(1)}_{[0,N]}=\alpha_1\left[S^{(0)}_{[0,N]}\left(1-\frac{2S^{(0)}_{[0,N]}}{1-\hat{r}_{[0,N]}(\lambda)}\right)+(S^{(0)}_{[0,N]})^2 \times\sum_{k=1}^{N-1}\left(1-\frac{1}{1-\hat{r}_{[k,0]}(\lambda)}-\frac{1}{1-\hat{r}_{[k,N]}(\lambda)}\right)\right].$$

By symmetry, we can rewrite this formula as

$$S^{(1)}_{[0,N]}=\alpha_1\left[S^{(0)}_{[0,N]}[1+(N-1)S^{(0)}_{[0,N]}]-2(S^{(0)}_{[0,N]})^2 \times\sum_{k=1}^{N}\left(1-\frac{1}{1-\hat{r}_{[0,k]}(\lambda)}\right)\right], \tag{C21}$$

which using Eq. (3.3) for $S^{(0)}_{[0,N]}$ is

$$S^{(1)}_{[0,N]}=\alpha_1\left[\frac{NS^{(0)}_{[0,N]}}{N-(N-1)S}-\frac{2S^{(0)}_{[0,N]}S}{N-(N-1)S}\sum_{k=1}^{N}\frac{1}{1-\hat{r}_{[0,k]}(\lambda)}\right]. \tag{C22}$$

From Eq. (C22) it is straightforward to derive the results of Eqs. (3.4)–(3.7).

## APPENDIX D: PROOF OF STOCHASTIC RESONANCE

### 1. An upper bound for the quantity $\hat{r}_{[0,k]}$

We consider an interval $[a,b]$ that is composed of $[a,c]$ and $[c,b]$. In $[a,c]$ and $[c,b]$ we have two stochastic processes and we denote $s_{ca}(t)dt$ the probability that starting from $a$ at time 0, one leaves $[a,c]$ through $c$ between times $t$ and $t+dt$ and $r_{ca}(t)dt$ the probability that starting from $a$ at time 0 one leaves $[a,c]$ through $a$ between times $t$ and $t+dt$. $s_{bc}$ and $r_{bc}$ denote the same quantities relative to the interval $[c,b]$ and $s_{ba},r_{ba}$ are the quantities relative to the entire interval $[a,b]=[a,c]\cup[c,b]$ for the joint stochastic process in this full interval (see Fig. 3). We denote the Laplace transform of a function $s$ by $\hat{s}$,

$$\hat{s}_{ca}(\lambda)=\int_0^\infty e^{-\lambda t}s_{ca}(t)dt.$$

Finally, we can also introduce the quantity $s_{ac}(t)dt$, which is the probability starting from $c$ to leave $[a,c]$ through

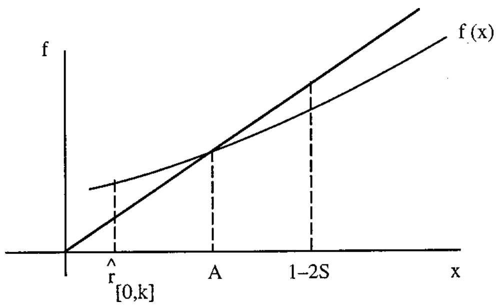


FIG. 4. Function $f(x)$ as defined by Eq. (D7), its fixed point, and the position of $1-2S$ if Eq. (D10) holds, as well as the position of $\hat{r}_{[0,k]}$.

$a$ for the first time in $[t,t+dt]$ and also $r_{ac}, s_{cb}, \ldots$ and their Laplace transforms. Then it is easy to prove that $\hat{s}_{ba}$ and $\hat{r}_{ba}$ are given by

$$\hat{s}_{ba}=\hat{s}_{bc}(1-\hat{r}_{ac}\hat{r}_{bc})^{-1}\hat{s}_{ca},$$
$$\hat{r}_{ba}=\hat{r}_{ca}+\hat{s}_{ac}\hat{r}_{cb}(1-\hat{r}_{ac}\hat{r}_{bc})^{-1}\hat{s}_{ac}. \quad \text{(D1)}$$

We write

$$y_{ba}=\hat{s}_{ba}^{-1}, \quad t_{ba}=\hat{r}_{ba}\hat{s}_{ba}^{-1}. \quad \text{(D2)}$$

Using these variables, the system (D1) can be written as

$$y_{ba}=\hat{s}_{ba}^{-1}y_{bc}-(\hat{s}_{ca}^{-1}\hat{r}_{ac})t_{bc},$$
$$t_{ba}=(\hat{r}_{ca}\hat{s}_{ca}^{-1})y_{bc}+\hat{s}_{ca}(1-\hat{s}_{ac}^{-1}\hat{r}_{ca}\hat{s}_{ca}^{-1}\hat{r}_{ac})t_{bc}. \quad \text{(D3)}$$

If the stochastic processes are symmetric in each interval so that

$$s_{ca}=s_{ac}, \quad r_{ca}=r_{ac},$$

etc., the system (D3) reduces to

$$y_{ba}=\hat{s}_{ca}^{-1}y_{bc}-(\hat{s}_{ca}^{-1}\hat{r}_{ca})t_{bc},$$
$$t_{ba}=(\hat{r}_{ca}\hat{s}_{ca}^{-1})y_{bc}+\hat{s}_{ca}(1-\hat{s}_{ca}^{-2}\hat{r}_{ca}^{2})t_{bc}. \quad \text{(D4)}$$

We now apply this result to the system of $N$ intervals with

$$b=0, \quad c=k, \quad a=k+1$$

and simply denote

$$y_k=y_{0k}, \quad t_k=t_{0k}$$

so that the system (D4) becomes in our case

$$y_{k+1}=\hat{s}^{-1}y_k-\hat{s}^{-1}\hat{r}t_k,$$
$$t_{k+1}=\hat{s}^{-1}\hat{r}y_k+\hat{s}(1-\hat{s}^{-2}\hat{r}^2)t_k, \quad \text{(D5)}$$

which allows us to recover $\hat{r}_{[0,k+1]}$ as

$$\hat{r}_{[0,k+1]}=\frac{t_{k+1}}{y_{k+1}}=\frac{\hat{s}^{-1}\hat{r}+\hat{s}(1-\hat{s}^{-2}\hat{r}^2)\hat{r}_{[0,k]}}{\hat{s}^{-1}-(\hat{s}^{-1}\hat{r})\hat{r}_{[0,k]}}. \quad \text{(D6)}$$

We now introduce the function (see Fig. 4)

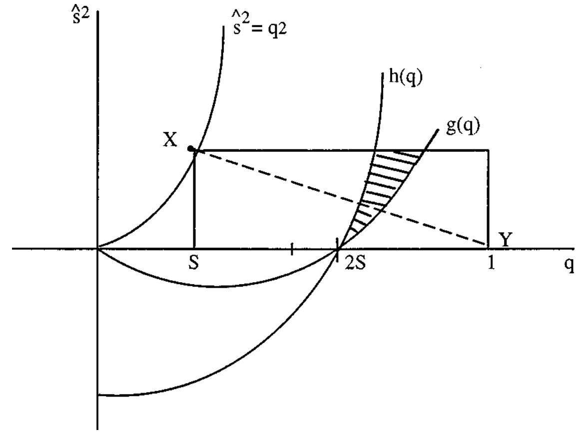


FIG. 5. Small curvilinear triangle where the inequalities (D-11) hold. The dotted time is the line followed by $[q(\lambda), \hat{s}^2(\lambda)]$ when $\lambda$ runs from 0 to $\infty$. It cuts this triangle.

$$f(x)=\frac{\hat{r}+\hat{s}^2(1-\hat{s}^{-2}\hat{r}^2)x}{1-\hat{r}x} \quad \text{(D7)}$$

so that $f$ is convex and increasing. Moreover, from Eqs. (D6) and (D7)

$$f(\hat{r}_{[0,k]})=\hat{r}_{[0,k+1]}.$$

It is trivial that $\hat{r}_{[0,k+1]}>\hat{r}_{[0,k]}$. Let us denote by $A$ the fixed point of $f$,

$$f(A)=A. \quad \text{(D8)}$$

We have a situation like that in Fig. 5, so that

$$\hat{r}_{[0,k]}<A. \quad \text{(D9)}$$

The inequality (4.15) is proved provided inequality (4.20) holds, namely,

$$A<1-2S. \quad \text{(D10)}$$

However, because $f(A)=A$ and $f$ is below the function $i(x)=x$ for $x>A$, in order that Eq. (D10) holds, it is sufficient that

$$f(1-2S)<1-2S,$$

which, using the definition (D7) of $f(x)$, gives us the inequality (4.20).

### 2. Discussion of the system of inequalities (4.40)–(4.44)

This system is

$$g(q)<\hat{s}^2<h(q),$$
$$\hat{s}^2\leqslant q^2, \quad \hat{s}^2\leqslant S^2, \quad S\leqslant q\leqslant 1, \quad \text{(D11)}$$

with

$$g(q)=q^2-2qS,$$
$$h(q)=q^2+\frac{4S^2}{1-2S}q-\frac{4S^2}{1-2S}. \quad \text{(D12)}$$

This system can be most simply discussed graphically. We represent this system in Fig. 5 with abscissa $q$ and ordinate $\hat{s}^2$. Both parabolas $\hat{s}^2=g(q)$ and $h(q)$ cut the axis at $q=2S$ and $2S<1$. The region where inequalities (D11) hold is shown in Fig. 5. The point $X=(q=S,\hat{s}^2=S^2)$ corresponds to $\lambda=0$, where $\hat{r}(0)=R$, $\hat{s}(0)=S$, and $q=1-R=S$. The point $Y=(q=1,\hat{s}^2=0)$, corresponds to $\lambda=\infty$, where $\hat{r}(\infty)=\hat{s}(\infty)=0$. When $\lambda$ varies from 0 to $\infty$, the point $(q(\lambda)=1-\hat{r}(\lambda),\hat{s}^2(\lambda))$ follows a trajectory joining $X$ to $Y$. This trajectory cuts out the allowed region on a certain interval corresponding to an interval $\Lambda$ of frequency $\lambda$ in which inequalities (4.13) and (4.14) hold and in which the stochastic resonance occurs.


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